THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280A Introductory Probability 2024-2025 Term 1
Homework Assignment 6
Due Date: 30 November, 2024 (Saturday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ______________________ Date ______________________

General Regulations

- All assignments will be submitted and graded on CUHK Blackboard. You can view your grades and submit regrade requests here as well.
- Late assignments will receive a grade of 0.
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction.
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. A set of 1000 cards numbered 1 through 1000 is randomly distributed among 1000 people with each receiving one card. Compute the expected number of cards that are given to people whose age matches the number on the card.

2. For a group of 100 people, compute

   (a) the expected number of days of the year that are birthdays of exactly 3 people;

   (b) the expected number of distinct birthdays.

3. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   Compute $\mathrm{Cov}(X, Y)$.

4. A fair die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a 6 and a 5. Find

   (a) $E[X]$.

   (b) $E[X|Y = 1]$.

   (c) $E[X|Y = 5]$.

5. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y},\ 0 < x < \infty,\ 0 < y < \infty$$

   Compute $E[X^2|Y = y]$.

6. The moment generating function of X is given by

$$M_X(t) = \exp\left(2e^t - 2\right)$$

   and that of Y by

$$M_Y(t) = \left(\frac{3}{4}e^t + \frac{1}{4}\right)^{10}.$$

   If X and Y are independent, what are

   (a) $P(X + Y = 2)$?

   (b) $P(XY = 0)$?

   (c) $E[XY]$?

7. Let $X_1, X_2, \cdots$ be a sequence of independent random variables having the probability mass function

$$P(X_n = 0) = P(X_n = 2) = 1/2,\ n \ge 1$$

   The random variable

$$X = \sum_{n=1}^{\infty} \frac{X_n}{3^n}$$

   is said to have the Cantor distribution. Find $E[X]$ and $\mathrm{Var}(X)$.

8. Suppose that $X$is a random variable with mean and variance both equal to 20. What can be said about $P\{0 < X < 40\}$?

9. Let $X_1, \cdots, X_{20}$ be independent Poisson random variables with mean 1.

   (a) Use the Markov inequality to obtain a bound on

$$P\left\{\sum_1^{20} X_i > 15\right\}.$$

   (b) Use the central limit theorem to approximate

$$P\left\{\sum_1^{20} X_i > 15\right\}.$$

10. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that

$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} \leq X_N.$$

That is, N is the point at which the sequence stops decreasing. Show that $E[N] = e$.